Right Noetherian Semigroups and Finite Generation

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Definitions

A semigroup S is *weakly right noetherian* if every right ideal of S is finitely generated.

S is *right noetherian* if every right congruence on S is finitely generated.

There are standard equivalent formulations in terms of ascending chain conditions and maximal conditions.

A semigroup is weakly right noetherian iff it contains no infinite ascending chain or infinite antichain of principal right ideals.

Right noetherian semigroups are weakly right noetherian.

Finite Generation: Old Results

Open Problem. Is every right noetherian semigroup finitely generated?

Theorem (Budach, Rédei). A commutative semigroup is (right) noetherian if and only if it is finitely generated.

Theorems (Hotzel/Kozhukhov). Let S be a right noetherian semigroup. Then S is finitely generated if any of the following conditions hold.
(1) Every $\mathcal{J}$-class of S is the class of some congruence on S.
(2) S satisfies the descending chain conditon on $\mathcal{J}$-classes.
(3) S is weakly periodic, i.e. for every $a \in S$ there exists $n \in \mathbb{N}$ such that $(S^1 a^n S^1)^2 = S^1 a^n S^1$.

Theorem (Kozhukhov). A semigroup that is both right noetherian and left noetherian is finitely generated.

Theorem (Kozhukhov). If there exists a non-finitely generated right noetherian semigroup, then there exists a countable such semigroup.

Finite Generation

Proposition. Any right noetherian semigroup S is generated by the union of a finite set and a set of representatives of the $\mathcal{L}$-classes of S.

Corollary. Let S be a right noetherian semigroup. If there exists a finitely generated subsemigroup T of S that intersects every $\mathcal{L}$-class of S, then S is finitely generated.

Corollary. A right noetherian semigroup is countable if and only if it has countably many $\mathcal{L}$-classes.

Inverse Semigroups

A group G is right noetherian if and only if all its subgroups are finitely generated, i.e. G is *noetherian*.

Proposition (Hotzel). Let G be a subgroup of a semigroup S. Then the lattice of subgroups of G embeds into the lattice of right congruences of S. Consequently, if S is right noetherian then so is G.

Lemma. A subsemilattice of a right noetherian semigroup is finite.

Theorem (Kozhukhov). For an inverse semigroup S, TFAE:

- S is right noetherian;
- $E(S)$ is finite and every subgroup of S is noetherian;
- every inverse subsemigroup of S is finitely generated.

Corollary. An inverse subsemigroup of a right noetherian semigroup is itself right noetherian.

Green's Relations, Units and Idempotents

The set of *right/left/two-sided units* of a monoid S is $R_1/L_1/J_1$. J_1 is a group, called the *group of units* of S.

Lemma. In a right noetherian semigroup S, we have $R_1 = L_1 = J_1$; equivalently, $S \setminus J_1$ is an ideal.

Proposition (Kozhukhov). Let S be a right noetherian semigroup. Then every $\mathcal{J}$-class of S is a finite union of incomparable $\mathcal{R}$-classes.

A semigroup is *right stable* if distinct $\mathcal{R}$-classes belonging to the same $\mathcal{J}$-class are incomparable.
Right noetherian semigroups are right stable.

Proposition. Let S be a right noetherian semigroup. Then $E(S)$ is contained in a finite union of $\mathcal{R}$-classes of S. In particular, S has only finitely many pairwise non-isomorphic maximal subgroups.

Green's Relations and Subgroups in Cancellative Monoids

A semigroup S is **cancellative** if for all $a, b, c \in S$,

$$ac = bc \Rightarrow a = b \quad \text{and} \quad ca = cb \Rightarrow a = b.$$

Let S be a cancellative monoid and let G denote its group of units. For any $a \in S$ we have

$$L_a = Ga, \quad R_a = aG, \quad H_a = Ga \cap aG \quad \text{and} \quad D_a = GaG.$$

We have transitive left and right actions of G on the sets

$$L_a/\mathcal{H} = \{H_{ga} : g \in G\} \quad \text{and} \quad R_a/\mathcal{H} = \{H_{ag} : g \in G\},$$

respectively. Let $\Gamma_l(H_a)$ and $\Gamma_r(H_a)$ denote the respective stabiliser subgroups of G with respect to H_a under these actions.

We have $H_a = \Gamma_l(H_a)a = a\Gamma_r(H_a)$ and $\Gamma_l(H_a) \cong \Gamma_r(H_a)$.

By the orbit-stabiliser theorem,

$$|G : \Gamma_l(H_a)| = |L_a/\mathcal{H}| \quad \text{and} \quad |G : \Gamma_r(H_a)| = |R_a/\mathcal{H}|.$$

When are $\mathcal{J}$-classes Congruence Classes?

Lemma. Let S be a weakly right noetherian cancellative monoid. Then every $\mathcal{J}$-class of S is a finite union of incomparable $\mathcal{R}$-classes. In particular, S is right stable.

Lemma. Let S be a right stable cancellative monoid, and let J be a $\mathcal{J}$-class of S. Then J is a class of some right congruence on S. Moreover, J is a class of some congruence on S if and only if it is a class of some left congruence on S.

Corollary. Let S be a right noetherian cancellative monoid such that $\mathcal{J} = \mathcal{R}$. Then every J-class of S is a class of some congruence on S, and hence S is finitely generated.

Corollary. Let S be a right noetherian, left stable, cancellative monoid. Then every J-class of S is a class of some congruence on S, and hence S is finitely generated.

Conditions Implying Finite Generation

Proposition. Let S be a weakly right noetherian cancellative monoid. Let G denote the group of units of S, let X be a finite set such that $S \setminus G = XS^1$, and let $T = \langle G \cup X \rangle$. If $S \neq T$, then there exist $a \in S \setminus T$ and $g \in G$ such that there is an infinite chain

$$\cdots >_{\mathcal{H}} ag^{-2} >_{\mathcal{H}} ag^{-1} >_{\mathcal{H}} a >_{\mathcal{H}} ag >_{\mathcal{H}} ag^2 >_{\mathcal{H}} \cdots$$

of elements belonging to R_a.

Corollary. Let S be a weakly right noetherian cancellative monoid such that:
(1) the group of units of S is finitely generated; and
(2) S satisfies either the minimal condition or maximal condition on $\mathcal{H}$-classes within an $\mathcal{R}$-class.
Then S is finitely generated.

(1) holds if S is right noetherian.

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of elements belonging to R_a.

Corollary. Let S be a weakly right noetherian cancellative monoid such that:
(1) the group of units of S is finitely generated; and
(2) there are only finitely many $\mathcal{J}$-classes above any $\mathcal{J}$-class.
Then S is finitely generated.

Proof. Suppose not, so that $S \neq T$. Then there exists $s \in S \setminus G$ such that $ag = sa$ (so $ag^n = s^n a$ for all $n \in \mathbb{N}$). Then $s >_{\mathcal{J}} s^2 >_{\mathcal{J}} \cdots >_{\mathcal{J}} s^n >_{\mathcal{J}} \cdots >_{\mathcal{J}} a$.

Conditions Implying Finite Generation

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of elements belonging to R_a.

Corollary. Let S be a weakly right noetherian cancellative monoid. If the group of units of S is finitely generated and periodic, then S is finitely generated.

Corollary. Let S be a right noetherian cancellative monoid. If the group of units of S is periodic, then S is finitely generated.

Conditions Implying Finite Generation

Proposition. Let S be a weakly right noetherian cancellative monoid. Let G denote the group of units of S, let X be a finite set such that $S \setminus G = XS^1$, and let $T = \langle G \cup X \rangle$. If $S \neq T$, then there exist $a \in S \setminus T$ and $g \in G$ such that there is an infinite chain

$$\cdots >_{\mathcal{H}} ag^{-2} >_{\mathcal{H}} ag^{-1} >_{\mathcal{H}} a >_{\mathcal{H}} ag >_{\mathcal{H}} ag^2 >_{\mathcal{H}} \cdots$$

of elements belonging to R_a. Moreover, the subgroup $\Gamma_l(H_a)$ of G has finite index in G, and the subgroup $\Gamma_r(H_a)$ of G has infinite index in G.

Corollary. Let S be a weakly right noetherian cancellative monoid. If the group of units of S is finitely generated and does *not* contain two isomorphic subgroups where one has finite index and the other infinite index, then S is finitely generated.

Virtually Polycyclic Groups

A group G is **polycyclic** if it has a subnormal series

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

such that each G_i/G_{i-1} is cyclic ($i \in \{1, \ldots, n\}$).
A group is **virtually polycyclic** if it has a polycyclic normal subgroup of finite index.
A group is (virtually) polycyclic iff it is noetherian and (virtually) solvable.

The *Hirsch length* of a virtually polycyclic group G, denoted $h(G)$, is the number of infinite factors in a polycyclic series of any polycyclic normal subgroup of finite index in G.

Fact: For $H \leq G$, we have $h(H) \leq h(G)$, and $h(H) = h(G) \Leftrightarrow |G : H| < \infty$.

Proposition. Let S be a weakly right noetherian cancellative monoid. If the group of units of S is virtually polycyclic, then S is finitely generated.

Other Noetherian Groups

Baer conjectured that a group is noetherian if and only if it is virtually polycyclic.

Ol'shanskii refuted this conjecture by constructing a simple 2-generated group G in which every proper subgroup is isomorphic to $\mathbb{Z}$.

If a weakly right noetherian cancellative monoid S has G as its group of units, then S is finitely generated.
Indeed, the only subgroup of finite index in G is G itself, and G is not $\mathbb{Z}$.

Tarski monster groups are noetherian and periodic.

As far as we know, there are no known noetherian groups that can be the group of units of a (weakly) right noetherian cancellative monoid that is not finitely generated.

Groups of Right Quotients

A *group of right quotients* of a (necessarily cancellative) semigroup S is a group G containing S such that $G = \{ab^{-1} : a, b \in S\}$.

A semigroup S is *left reversible* if any two principal right ideals of S intersect: $aS \cap bS \neq \varnothing$ for all $a, b \in S$.

Theorem (Dubreil). A cancellative semigroup can be embedded in a group of right quotients if and only if it is left reversible.

Groups of Right Quotients

Proposition. Every weakly right noetherian cancellative monoid is left reversible (and hence embeds into a group of right quotients).

For a group G, denote its lattice of subgroups by $\mathrm{Sub}(G)$.
For a monoid S, denote its lattice of right congruences by $\mathrm{RCon}(S)$.

Proposition. Let S be a monoid that embeds into a group of right quotients K. Then the mapping

$$\mathrm{Sub}(K) \to \mathrm{RCon}(S),\ H \mapsto \rho_H = \{(a,b) \in S \times S : ab^{-1} \in H\}$$

is a lattice embedding.

Corollary. Every right noetherian cancellative monoid embeds into a noetherian group of right quotients (and is hence countable).

Groups of right quotients

Let S be a right noetherian cancellative monoid and let G be its group of units.

If the group K of right quotients S is periodic or virtually polycyclic, then so is G, and hence S is finitely generated.
If K is Olshanskii's noetherian group, then either $S = K$ (in which case S is finitely generated) or $G \cong \mathbb{Z}$ (since it is proper subgroup of K), and hence S is finitely generated.

Let $G = \langle X \mid R \rangle$, let $a \in S$, let $\theta : \Gamma_l(H_a) \to \Gamma_r(H_a)$ an isomorphism, and let G' denote the subgroup $\langle X \cup \{a\} \rangle$ of K. Then $\Gamma_l(H_a)$ and $\Gamma_r(H_a)$ are conjugate subgroups of G' (and hence of K), and hence have the same index in G' (and in K). Moreover, G' is a quotient of the HNN-extension

$$\langle X, a \mid R,\ a^{-1}ga = g\theta\,(g \in \Gamma_l(H_a)) \rangle$$

of G with respect to θ.

Cancellative Monoids: Open Problems

(1) Is every right noetherian cancellative monoid finitely generated?

(2) Is every weakly right noetherian cancellative monoid with a *noetherian* group of units finitely generated?

(3) Is every weakly right noetherian cancellative monoid with a *finitely generated* group of units finitely generated?